Portfolio Risk Decomposition (and Risk Budgeting)

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R-Squared Risk Management

Introduction to Risk Decomposition

- Active managers take risk in the expectation of achieving outperformance of their benchmark

- Mandates often stipulate the amount of Tracking Error (risk relative to the benchmark) allowed

- However, two other matters are at least as important
  - Having the right <u>**amounts**</u> of risk (Risk decomposition by holdings)
  - Having the right <u>**kinds**</u> of risk (Risk decomposition by factors)

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Risk Decomposition by Factors

- The type of risk being taken should correspond to the way in which the manager expects to outperform
  - A stock picker **should not** be taking currency risk
  - Market-timers **should** be managing their market beta
  - Sector rotators **should** be managing their sector factor exposures (not just their holding sizes)

- Risk Decomposition by factors clarifies **what kinds** of risk there are, and **how big** the factor bets are relative to each other, and to stock specific risk

Risk Attribution by Factors

- There is an important distinction to be made between Risk Decomposition using the factors in a given risk model, and Risk Attribution
- Risk Attribution is used to determining the exposure of a portfolio to factors that are **not** in the risk model
- For example, many systems give the Beta of the portfolio relative to its benchmark, which is rarely a factor in the risk model
- Statistical risk models use Risk Attribution to relate the portfolio risk to macro-economic and other factors

Risk Decomposition by Holdings

- This talk focuses on Risk Decomposition by holdings

- We examine two cases :-
  - Decomposition of Absolute Risk
  - Decomposition of Relative Risk or Tracking Error

- In each case, we want to answer two questions:
  - How much risk is coming from each holding ?
  - How would the portfolio risk be affected by a small change in each holding (the marginal contribution to risk) ?

A Sneak Peek Ahead

- This kind of Risk Decomposition forms the basis for Risk Budgeting decisions
- We will be looking at the Actual, Percentage and Marginal Contributions to Portfolio Risk, both from individual holdings and from groups of holdings
- In the case of Absolute Risk Decomposition, there is a unique answer to these questions
- Unfortunately for Risk Budgeting, there **isn't** a unique answer in the case of Tracking Error Decomposition

The Simplest Possible Example

- Before we get into the fancy algebra, let's consider a very simple example

- Our benchmark is that old pension fund favourite :
  - 60% stocks, 40% bonds

- The manager is nervous about stocks and neutral on bonds, so our portfolio consists of :
  - 50% stocks, 40% bonds, 10% cash

- The Risk Decomposition should be simple …

Portfolio Holdings data

	Portfolio	Benchmark	Difference
	x(i)	b(i)	d(i)
Totals =	100%	100%	0%
Stocks =	**50%**	**60%**	-10%
Bonds =	**40%**	**40%**	0%
Cash =	**10%**	**0%**	10%

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Absolute Volatilities & Correlations

Absolute Volatility & Correlation Matrix			
	Stocks	Bonds	Cash
Std. dev.	**15.00**	**8.00**	**0.00**
15.00	1.00	**0.40**	**0.00**
8.00	0.40	1.00	**0.00**
0.00	0.00	0.00	1.00

The Absolute Covariance Matrix !

Absolute Covariance Matrix		
Stocks	Bonds	Cash
225.00	48.00	0.00
48.00	64.00	0.00
0.00	0.00	0.00

The Algebra of Risk Decomposition

- We begin by breaking down the total variance of a portfolio into contributions from individual holdings

- We have $$V_P = \sum_{i}^{N} \sum_{j}^{N} x_i x_j C_{ij}$$

- From which we derive individual contributions to variance as

$$ACV_{iP} = \sum_{j}^{N} x_i x_j C_{ij}$$

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Actual Risk Contribution Simplified

$$ACV_{iP} = \sum_{j}^{N} x_i x_j C_{ij} = x_i \sum_{j}^{N} x_j C_{ij}$$

$$= x_i \sum_{j}^{N} x_j \, cov\left(R_i, R_j\right) = x_i \, cov\left(R_i, \sum_{j}^{N} x_j R_j\right)$$

$$= x_i \, cov\left(R_i, R_P\right) = x_i C_{iP}$$

where C_{ip} is the covariance of asset i with the portfolio P.

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Percentage Risk Contribution

- The conversion of Actual Contributions to Risk to Percentage Contributions is amazingly simple:

$$PCV_{iP}\% = \frac{ACV_{iP}}{V_P} * 100\,\%$$

- We simply divide the Actual Contribution by the Total Risk and multiply by 100

The Nitty Gritty Calculation Details

Cov(P,B) = 98.86	Covariance(asset, portfolio)		
	Stocks	Bonds	Cash
Stocks =	112.50	24.00	0.00
Bonds =	19.20	25.60	0.00
Cash =	0.00	0.00	0.00
Cov(i,P) =	**131.70**	**49.60**	**0.00**
Corr(i,P) =	**0.948**	**0.670**	**0.000**

Absolute Risk Decomposition

Absolute Portfolio Variance			
V(p) =	85.69	SD(p) =	9.26
Stocks =	56.25	9.60	0.00
Bonds =	9.60	10.24	0.00
Cash =	0.00	0.00	0.00
ACV(i) =	**65.85**	**19.84**	**0.00**
PCV(i) =	**77%**	**23%**	**0%**

Contributions from Groups of Holdings

- We can generalise these expressions from individual holdings to groups of holdings as follows :-

$$ACV_{Energy} = \sum_{i \in Energy} ACV_{iP}$$

$$PCV_{Energy}\% = \sum_{i \in Energy} PCV_{iP}\%$$

Marginal Contribution to Variance

- For the total portfolio risk we have :-

$$V_P = \sum_i^N \sum_j^N x_i x_j C_{ij}$$

- The Marginal Contribution to Variance is defined as :-

$$MCV_{iP} = \frac{\partial V_P}{\partial x_i} = 2\sum_j^N x_j C_{ij} = 2C_{iP}$$

which really couldn't be simpler!

Marginal Contribution to Risk (S.D.)

- Bearing in mind that : $V_p = S_p^2$

  we have : $\frac{\partial V_P}{\partial S_P} = 2S_P$

- And so, trivially,

$$MCR_{iP} = \frac{\partial S_P}{\partial x_i} = \frac{\partial S_P}{\partial V_P}\frac{\partial V_P}{\partial x_i} = \frac{\frac{\partial V_P}{\partial x_i}}{\frac{\partial V_P}{\partial S_P}} = \frac{MCV_{iP}}{2S_P} = \frac{2C_{iP}}{2S_P} = \frac{C_{iP}}{S_P}$$

Marginal Contributions to Risk

Marginal Contributions to Portfolio Variance		
Stocks	MCV(i) =	2.634
Bonds	MCV(i) =	0.992
Cash	MCV(i) =	0.000

Marginal Contributions to Portfolio Risk		
Stocks	MCR(i) =	0.142
Bonds	MCR(i) =	0.054
Cash	MCR(i) =	0.000

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Summary of Absolute Decomposition

Absolute Risk Decomposition by Holdings - Portfolio												
	Formulae =				x(i) * C(i,P)	100*ACV(i) / V(p)	2 * C(i,P) / 100	(C(i,P)/100) / (S(p))		C(i,P) / (S(i)*S(p))	PCV(i) / x(i)	C(i,P) / V(p)
Holding	x(i)	S(i)	C(i,P)		= ACV(i)	= PCV(i)	= MCV(i)	= MCR(i)		= Corr(i,P)	= Beta(i,P)	
Stocks	50%	15.00	131.70		65.85	77%	2.634	0.142		0.948	1.54	1.54
Bonds	40%	8.00	49.60		19.84	23%	0.992	0.054		0.670	0.58	0.58
Cash	10%	0.00	0.00		0.00	0%	0.000	0.000		0.000	0.00	0.00
Portfolio	100%				85.69	100%					1.00	1.00
Portfolio variance =			V(p) =		85.69		Portfolio risk (s.d.) =				S(p) =	9.26

Tracking Error Decomposition

- Tracking error is the variance of relative returns
- Relative returns are defined as follows :-

$$\hat{R}_P = R_P - R_B$$

where $\hat{R}_P$ is the relative return on the portfolio,

R_P is the absolute return on the portfolio, and

R_B is the absolute return on the benchmark

Here's the Important Bit !!

- Portfolio and benchmark returns are defined as:

$$R_p = \sum_i^N x_i R_i \qquad\qquad R_B = \sum_i^N b_i R_i$$

- So relative returns can be defined as :-

$$\hat{R}_P = \sum_i^N (x_i - b_i) R_i$$

$$= \sum_i^N x_i (R_i - R_B) = \sum_i^N (x_i - b_i)(R_i - R_B)$$

Three Definitions of Tracking Error

- Corresponding to each of these formulations, we get three different expressions for Tracking Variance :-

$$TV_P = \sum_i^N \sum_j^N (x_i - b_i)(x_j - b_j) C_{ij}$$

$$= \sum_i^N \sum_j^N x_i x_j \hat{C}_{ij}$$

$$= \sum_i^N \sum_j^N (x_i - b_i)(x_j - b_j) \hat{C}_{ij}$$

Three Versions of Tracking Error

- These three sets of equations for relative return and risk (T.E.) may be characterised as follows:
  - First uses relative holdings and absolute returns
  - Second uses absolute holdings and relative returns
  - Third uses relative holdings and relative returns

- It is very easy to demonstrate that these expressions are equivalent at ***the aggregate level***

- However, they lead to different decompositions

Relative Volatilities & Correlations

Relative Volatility & Correlation Matrix			
	Stocks	Bonds	Cash
Std. dev.	5.56	8.34	10.69
5.56	1.00	(1.00)	(0.67)
8.34	(1.00)	1.00	0.67
10.69	(0.67)	0.67	1.00

The Relative Covariance Matrix !

Relative Covariance Matrix		
Stocks	Bonds	Cash
30.88	(46.32)	(39.92)
(46.32)	69.48	59.88
(39.92)	59.88	114.28

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Tracking Error Decomposition - 1

Relative weights & Absolute covariances			
TV(p) =	**2.25**	**TE(p) =**	**1.50**
Stocks =	2.25	0.00	0.00
Bonds =	0.00	0.00	0.00
Cash =	0.00	0.00	0.00
ACV(i) =	**2.25**	**0.00**	**0.00**
PCV(i) =	**100.0%**	**0.0%**	**0.0%**

Tracking Error Decomposition - 2

Absolute weights & Relative covariances			
TV(p) =	**2.25**	**TE(p) =**	**1.50**
Stocks =	7.72	(9.26)	(2.00)
Bonds =	(9.26)	11.12	2.40
Cash =	(2.00)	2.40	1.14
ACV(i) =	**(3.54)**	**4.25**	**1.54**
PCV(i) =	**-157.3%**	**188.8%**	**68.5%**

Tracking Error Decomposition - 3

Relative weights & Relative covariances			
TV(p) =	**2.25**	**TE(p) =**	**1.50**
Stocks =	0.31	0.00	0.40
Bonds =	0.00	0.00	0.00
Cash =	0.40	0.00	1.14
ACV(i) =	**0.71**	**0.00**	**1.54**
PCV(i) =	**31.5%**	**0.0%**	**68.5%**

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Comments on TE Decompositions

- The first, using Absolute Covariances, treats cash as riskless, and so attributes <u>**all**</u> the risk to the stock bet
- The second uses Absolute Holdings, and attributes most of the risk to the neutral position in bonds !!
- This second version will also say that a zero holding in the portfolio has no contribution to Tracking Error, even if the asset is held in the benchmark
- The third version is (usually) the most intuitive, and **in this case gives a very sensible answer**

Marginal Contributions to TE - 1

- Changing an Absolute Holding by a small amount is the same as changing a Relative Holding by a small amount, since the Benchmark Holding is fixed

- Thus, we only get **two** different sets of results for the Marginal contributions to Tracking Error

- The first version is given by :-

$$MCVRA_i = \frac{\partial TV_P}{\partial x_i} = \frac{\partial TV_P}{\partial d_i} = 2\sum_j d_j C_{ij} = 2\left(C_{iP} - C_{iB}\right)$$

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Marginal Contributions to TE - 2 & 3

- The second and third versions are given by :-

$$MCVAR_i = \frac{\partial TV_P}{\partial x_i} = 2\sum_j^N x_j \hat{C}_{ij} = 2\hat{C}_{iP}$$

$$MCVRR_i = \frac{\partial TV_P}{\partial d_i} = 2\sum_j^N d_j \hat{C}_{ij} = 2\hat{C}_{iP}$$

which are the same.

Marginal Contributions to TE - 1

Marginal Contributions to Tracking Error		
using Absolute Covariance matrix		
Stocks	MCR(i) =	(0.150)
Bonds	MCR(i) =	(0.032)
Cash	MCR(i) =	0.000

Marginal Contributions to TE - 2 & 3

Marginal Contributions to Tracking Error		
using Relative Covariance matrix		
Stocks	MCR(i) =	(0.047)
Bonds	MCR(i) =	0.071
Cash	MCR(i) =	0.103

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Comment on Marginal Contributions

- At first sight these results seem contradictory
  - The first version says if we increase the bond holding, the TE will <u>decrease</u>
  - The second and third versions say if we increase the bond holding, the TE will <u>increase</u>

- However, note that we still have the budget constraint, so we still get the same net result

- In the first case : -0.032 - (-0.150) = **0.118**
- In the other cases : 0.071 - (-0.047) = **0.118**

Conclusions

- Most institutional investors are concerned with Tracking Error rather than Absolute Risk
- Risk Decomposition is at least as important as the overall level of risk in an actively-managed portfolio
- However, there are different answers!
- The industry default is the first decomposition, but this is almost certainly not the best, or most intuitive
- Caveat Manager !!

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